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INVERSE PROBLEMS CONNECTED WITH TWO-POINT BOUNDARY VALUE PROBLEMS*

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1. Introduction. We shall study the following nonlinear two-point boundary value problem:

$$(1.1)\qquad \begin{gathered} u'' + \lambda f(u) = 0, \qquad 0 < x < 1, \\ u'(0) = u(1) = 0. \end{gathered}$$

Under certain conditions on f we seek those values of $\lambda > 0$ for which positive solutions exist, and, in particular, we are interested in bifurcation values of λ and the number of solutions corresponding to a given λ.

This problem was motivated by recent studies [1]–[6] of nonlinear eigenvalue problems of the form

$$(1.2)\qquad \begin{aligned} Lu &= \lambda F(x,u), & x &\in D, \\ Bu &= 0, & x &\in \partial D, \end{aligned}$$

where L is a linear ordinary or elliptic partial differential operator, and B is a linear boundary operator. D and ∂D denote the domain and its boundary respectively. (The references [1]–[6] can be consulted for a precise formulation of the general situation (1.2) and also for physical applications giving rise to these problems.) These previous investigations of (1.2) have been concerned with questions of existence, uniqueness, multiplicity, stability, and bifurcation phenomena. It has become clear that certain quantities (and their geometrical interpretations) play a crucial role in the answers to these questions. Furthermore, as so often is the case in formulating and proving general theories, the proper critical quantities have been suggested or discovered by a rather complete analysis of certain special problems.

We define the multiplicity of a number λ as the number of distinct positive solutions of (1.1) corresponding to that value of λ. Our main goal is to present a series of special examples for the purpose of finding those properties of the nonlinearity $f(u)$ which control or change the multiplicity of a given λ. We have chosen the name "inverse" problem because our approach will be to specify the multiplicity and then construct the functions $f(u)$ giving rise to the specified situation. More precisely, if we define the maximum value of a solution as its amplitude, then for a given graph (called the response curve) of amplitude A versus λ we shall determine functions $f(u)$ yielding that response.

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In § 2 we first present certain general results. These include asymptotic descriptions of the response diagram for both large and small values of A and λ and some general results of a qualitative nature. Our main results are contained in § 3 where we construct functions $f(u)$ yielding specified response diagrams. The selection of the nonlinear functions $f(u)$ was motivated by research on some recently occurring problems [1]–[6] for which the multiplicity question is still open.

2. General results. In the following, $u = u(x)$ denotes a positive solution of the problem (1.1) and $A \equiv u(0)$.

Motivated by the physical examples in [1]–[6] we shall assume that the function $f(u)$ in (1.1) is positive and piecewise continuously differentiable in some interval $0 \leqq u < u_1$. It is easily seen from the differential equation in (1.1) that if $f(u) > 0$ for $0 \leqq u < u_1$ and $A < u_1$, then $u'(x) < 0$ for all $0 < x \leqq 1$ so that $u(x) < A$ for $0 < x \leqq 1$. On the other hand, if there exists an interval above u_1 where $f(u) \leqq 0$, and we assume that A is in this interval, then $u'(x)$ would be positive as long as $u > u_1$ so that $u(x)$ could never become less than u_1 and hence $u(x)$ could never satisfy $u(1) = 0$. Therefore, the positive solutions we are considering always assume values for which $f(u) > 0$.

Let

$$V(u) = \int_0^u f(u)\,du. \tag{2.1}$$

The differential equation in (1.1) then has the first integral

$$\tfrac{1}{2}u'^2 + \lambda V(u) = \text{const.} \tag{2.2}$$

For a solution of the problem (1.1) the constant is equal to $\lambda V(A)$ since $u'(0) = 0$. By integration, (2.1) then becomes

$$(2\lambda)^{1/2}x = \int_{u(x)}^{A} [V(A) - V(u)]^{-1/2}\,du. \tag{2.3}$$

Applying the boundary condition $u(1) = 0$ in (2.3) we obtain the relation

$$(2\lambda)^{1/2} = \int_0^A [V(A) - V(u)]^{-1/2}\,du \tag{2.4}$$

between the value of the parameter λ and the value of A for a solution of the problem (1.1) with that value of λ. We shall call the graph of this relationship the *response curve* corresponding to the function $f(u)$. If, for a particular value of λ, (2.4) has the solutions $A_1, \cdots, A_n$, then n different solutions of the problem exist and we shall say that the *multiplicity* of that value of λ is n.

Now, introduce V as the variable of integration in (2.4) and denote the inverse function of $V = V(u)$ by $u = \varphi(V)$. Since $V'(u) = f(u) > 0$ this function is always unique. Equation (2.4) then becomes

$$(2\lambda)^{1/2} = \int_0^z \frac{\varphi'(V)\,dV}{(z - V)^{1/2}}, \tag{2.5}$$

where $z = V(A)$. If $\lambda = \lambda(z)$ is a known function, (2.5) is an integral equation of

Abel's type. Its solution is [7, p. 158]

$$\varphi'(V) = \frac{1}{\pi}\int_0^V \frac{1}{(V-z)^{1/2}} \frac{d}{dz}(2\lambda)^{1/2}\, dz. \tag{2.6}$$

It is easily shown that if g is any sufficiently smooth function, then

$$\frac{d}{dV}\int_0^V \frac{g(z)}{(V-z)^{1/2}} dz = g(0)V^{-1/2} + \int_0^V \frac{g'(z)}{(V-z)^{1/2}} dz. \tag{2.7}$$

Formula (2.5) shows that $\lambda(0) = 0$. Therefore, using (2.7) and the fact that $\varphi(0) = 0$, we may integrate (2.6) to obtain

$$u \equiv \varphi(V) = \frac{1}{\pi}\int_0^V \left[\frac{2\lambda}{V-z}\right]^{1/2} dz. \tag{2.8}$$

This formula and

$$f \equiv 1/\varphi'(V), \tag{2.9}$$

which follows from (2.1) and the definition of φ, constitute the solution of the problem of finding a function $f(u)$ such that λ becomes a prescribed function of z. We shall call this problem the *inverse of the problem* (1.1). Formulas analogous to (2.8) and (2.9) have previously been obtained by M. Urabe [8, p. 259] in connection with inverse problems associated with periodic oscillations of autonomous systems.

It should be noted that the function $\lambda = \lambda(z)$ which here is assumed to be given is not the same as the function of which the response curve is the graph. The latter is given parametrically with z as the parameter by $A = \varphi(z)$ and $\lambda = \lambda(z)$. Since our main aim is to construct f-functions for which the response curves have intervals of prescribed multiplicity of λ, this fact is not serious, for such intervals will exist for the function $\lambda = \lambda(z(A))$ if they exist for the function $\lambda = \lambda(z)$.

The following physical interpretation of the problem (1.1) and its inverse is useful as a guide for its properties and for the construction of the examples in the next section. Let t denote dimensionless time and s the dimensionless arclength along a curve C with equation $y = y(s)$, where y is a Cartesian coordinate. If a particle of unit mass slides along C without friction in a gravitational field of unit strength directed along the negative y-axis, the equation of motion of the particle is

$$d^2s/dt^2 + y'(s) = 0. \tag{2.10}$$

If the particle starts at time $t = 0$ from rest at some point $s = A$, say, on C and reaches the point $s = 0$ at time $t = \lambda^{1/2}$, then $s'(0) = s(\lambda^{1/2}) = 0$. Therefore, if we introduce a new time variable $x = \lambda^{-1/2}t$ and define the functions $u = u(x)$ and $f = f(u)$ by $u(x) = s(t)$ and $f(u) = y'(s)$, then $u = u(x)$ satisfies the equation

$$d^2u/dx^2 + \lambda f(u) = 0 \tag{2.11}$$

and the boundary conditions $u'(0) = 0$ and $u(1) = 0$. That is, u is a solution of the problem (1.1). The "forward" problem, to find the roots A as functions of λ, may thus be interpreted as that of finding where a particle should start on C in order that it reach $s = 0$ at time $t = \lambda^{1/2}$. Since the function $V = V(u)$ defined in (2.1) in this interpretation is the potential energy or the height above the point $s = 0$, $z = V(A)$ is the vertical distance through which the particle falls. The inverse problem, to find a function $f = f(u)$ such that λ depends on z in a prescribed manner, may similarly be interpreted as that of shaping C so that the time of transversal becomes a given function of the height through which the particle falls. A special case of the inverse problem is Abel's classical problem of the tautochrone ($\lambda^{1/2}$ independent of z) and, indeed, the way we solve the inverse problem is just a generalization of the way in which Abel solved his.

The interpretation described above suggests several of the properties of the problem (1.1) and its inverse. For example, in view of this interpretation one would expect that if C flattens out with increasing y so that $y'' < 0$, i.e., $f' < 0$, then it is not possible for a particle which starts higher up to catch up with one starting from a lower level. This means that if f is monotone decreasing, the problem (1.1) should only have one positive solution for each $\lambda > 0$. In fact, that this is so is easily shown from (2.2). On the other hand, it is reasonable to expect that if C bends upward sufficiently fast, it should be possible for a particle which starts high up to catch up with one starting at some lower level. If this is true, the problem (1.1) should have at least two positive solutions for some values of λ. A precise result of this nature was recently found by T. Laetsch [6]. He proved that if $f(u) > 0$, $f'(u) > 0$, and $f''(u) > 0$ for all $u \geqq 0$ and if $f(u)/u \to \infty$ as $u \to \infty$, then there exists a positive number λ^* such that for λ in $0 < \lambda < \lambda^*$ the problem (1.1) has exactly two positive solutions, if $\lambda = \lambda^*$ it has exactly one positive solution and if $\lambda > \lambda^*$ it has no positive solutions. (Actually, Laetsch proved his result under weaker smoothness conditions than stated here.)

In order to obtain our results, we shall need a certain amount of general qualitative information about the shape of the response curve. First, we prove the following theorem.

THEOREM 1. *If*

$$\inf_{u \geqq 0} [f(u)/u] \equiv m_0 > 0, \tag{2.12}$$

then the problem (1.1) *cannot have a positive solution for all* $\lambda > 0$, *and, in fact, in order that such a solution exist* λ *must satisfy the inequality*

$$\lambda \leqq \mu/m_0, \tag{2.13}$$

where μ *is the smallest eigenvalue of*

$$\begin{gathered} v'' + \mu v = 0, \\ v'(0) = v(1) = 0. \end{gathered} \tag{2.14}$$

Proof. From the identity

$$\int_0^1 (vu'' - uv'')\, dx = \int_0^1 \frac{d}{dx}(vu' - uv')\, dx = 0$$

we obtain, using (1.1) and (2.14), that

$$\lambda \int_0^1 vf(u)\,dx = \mu \int_0^1 uv\,dx.$$

Thus,

$$\frac{\mu}{\lambda} = \frac{\displaystyle\int_0^1 \frac{f(u)}{u} uv\,dx}{\displaystyle\int_0^1 uv\,dx} \geqq \inf_{u \geqq 0} \left[\frac{f(u)}{u}\right] \equiv m_0$$

which proves (2.13). This result may also be found in [6].

We now present an immediate corollary of Theorem 1.

THEOREM 2. *A necessary condition that a positive solution of the problem* (1.1) *exist for all* $\lambda > 0$ *is that* $\lim_{u\to\infty} [f(u)/u]$ *exist and*

$$\lim_{u\to\infty} [f(u)/u] = 0.$$

Proof. If m_0 in (2.12) was positive, the set of λ's for which a positive solution exists would be bounded according to Theorem 1, which contradicts the hypothesis. Therefore, $m_0 = 0$. But for u bounded, $f(u)/u > 0$ since $f(u) > 0$. Thus, the minimum value, 0, of $f(u)/u$ must be attained at $u = \infty$, which proves the theorem.

In order to be able to construct functions $\lambda = \lambda(z)$ such that the corresponding functions $f = f(u)$ have certain qualitative properties, we need some asymptotic results which are now presented.

THEOREM 3. *If f can be expanded in a power series in some neighborhood of* $u = 0$:

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots \tag{2.15}$$

with $a_0 \neq 0$, *then*

$$(2\lambda)^{1/2} = z^{1/2}(b_0 + b_1 z + b_2 z^2 + \cdots), \tag{2.16}$$

in some neighborhood of $z = 0$, *and conversely. Here*

$$b_0 = 2/a_0 \quad \textit{and} \quad b_1 = -4a_1/3a_0^3. \tag{2.17}$$

Proof. Formula (2.16) is derived by inserting (2.15) in (2.1) after which the series obtained for $V = V(u)$ is inverted to give a series for $u = \varphi(V)$ and for $f^{-1} = \varphi'(V)$. When the latter is inserted in (2.5), the series (2.16) results. Similarly, (2.15) is derived from (2.16) by means of (2.8).

THEOREM 4. *If* $0 < a \leqq f(u)$ *for* $u \geqq 0$, *then* $2z/b^2 \leqq \lambda \leqq 2z/a^2$ *for* $z \geqq 0$.

Proof. Because of (2.9) and the hypothesis we have that $b^{-1} \leqq f^{-1} = \varphi'(V) \leqq a^{-1}$ for $V \geqq 0$. The result then follows immediately from (2.5).

In a similar way, the following two results are also obtained.

THEOREM 5. *If* $f(u) = b + O(u^{-\beta})$ *as* $u \to \infty$, *where* $b > 0$ *and* $0 < \beta < 1$ *are constants, then*

$$(2\lambda)^{1/2} = 2z^{1/2}/b + O(z^{\beta/(1+\beta)}) \qquad \textit{as } z \to \infty. \tag{2.18}$$

THEOREM 6. *If*

$$f(u) = k(c - u) + O((c - u)^2) \tag{2.19}$$

as $u \to c$, *where* c *is a positive constant, then*

$$(2\lambda)^{1/2} = -(2k)^{-1/2} \ln (V_0 - z) + \mu(z), \tag{2.20}$$

where $\mu(z)$ *is a continuous function of* z *in the interval* $0 < z \leqq V_0 = V(c)$, *and conversely.*

3. Examples.

Example 1. We shall demonstrate that if $f(u)$ is monotone increasing from $f(0) = 1$ to b as $u \to \infty$, then, depending on the value of b, the problem (1.1) may have exactly one positive solution for all values of $\lambda \geqq 0$ or there may exist intervals of λ with multiplicity up to three.

To choose the function $\lambda = \lambda(z)$ such that $f(u)$ behaves as specified, we use Theorems 3, 4 and 5. Since $f(0) = 1$, $(2\lambda)^{1/2}$ must behave as given in (2.16) with $b_0 = 2$ as $z \to 0$. Furthermore, $\lambda = \lambda(z)$ must be such that $2z/b^2 \leqq \lambda \leqq 2z$ for all $z \geqq 0$ with $\lambda(z)$ approaching $2z/b^2$ as $z \to \infty$. The function

$$\lambda = 2z[(1 + z)/(1 + bz)]^2 \tag{3.1}$$

satisfies all these requirements. A simple calculation shows that this function is monotone increasing if $b \leqq 9$, while it has exactly one positive maximum and one positive minimum point if $b > 9$. Thus, the problem (1.1) for which $\lambda = \lambda(z)$ is given by (3.1) has exactly one positive solution for all $\lambda \geqq 0$ if $b \leqq 9$, and, depending on the value of λ, either one, two, or three positive solutions if $b > 9$.

Inserting λ given by (3.1) in (2.8) and (2.9), we find that

$$u \equiv \varphi(V) = 2\frac{b - 1}{b^2}[1 - (1 + bV)^{-1/2}] + \frac{V}{b} \tag{3.2}$$

and

$$f \equiv \frac{1}{\varphi'(V)} = \frac{b}{(b - 1)(1 + bV)^{-3/2} + 1}. \tag{3.3}$$

The response curve of the function $f(u)$ given by (3.2) and (3.3) is given parametrically by (3.1) with $V = z$ and (3.2). The function $f(u)$ and the response curve are plotted in Figs. 1 and 2.

The general shape of Figs. 1 and 2 is clearly suggested by T. Laetsch [6], [9]. For functions $f(u)$ having the properties specified, D. S. Cohen and T. Laetsch [2] have proved that if $f(u)$ also satisfies

$$(d/du)[f(u)/u] < 0 \tag{3.4}$$

for all $u \geqq 0$, then the problem (1.1) cannot have more than one positive solution for any $\lambda \geqq 0$. The condition (3.4) has the following two interesting geometrical properties: (i) any line segment from the origin to the function lies below the graph of the function, and (ii) the tangent to the curve $y = f(u)$ on $u \geqq 0$ intersects the axis of ordinates (i.e., the $u = 0$ axis) in $y > 0$. Our present example (see Figs. 1 and 2) seems to indicate that it is when these conditions are violated that uniqueness is lost and multiple solutions appear. The value $b = 9$ is a bifurcation point in b; that is, $b = 9$ is the dividing point between the case when all

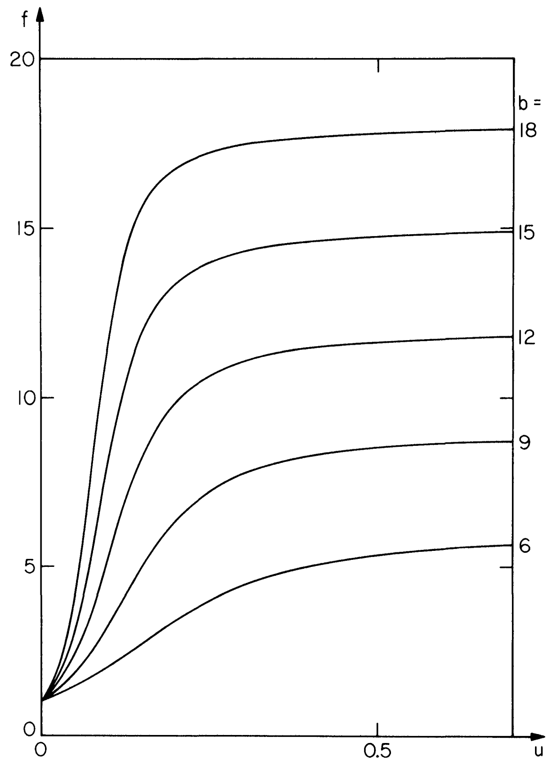

FIG. 1

$\lambda \geqq 0$ are of multiplicity one and the case when there exist values of λ with multiplicities one, two, three. For $b = 9$, numerical calculations show that the quantity

$$u^2 \frac{d}{du}\left[\frac{f(u)}{u}\right] = uf'(u) - f(u)$$

changes sign—thus that $(d/du)[f(u)/u]$ changes sign is a necessary condition for multiple solutions to exist. This has also been proved by Cohen and Laetsch [2]. It is not a sufficient condition; but all calculations indicate that it is a very good "approximate" sufficient condition.

Example 2. Boundary value problems of the form (1.1) occur in chemical reactor theory [2], [4]. Here $u = u(x)$ is the temperature distribution in the reactor, λ is a constant depending on the physical parameters, and $f(u)$ represents the rates of chemical production of the species (or, equivalently, the rate of heat generation) in the reactor. Since rates of production of the chemical species are completely determined by the temperature distribution, different solutions of the problem (1.1) correspond to different processes taking place in the reactor. Which of the solutions will occur in an actual process depends on how the reaction is started.

It has been observed experimentally that depending on λ, one, two, or three different temperature distributions are possible. In order to demonstrate that

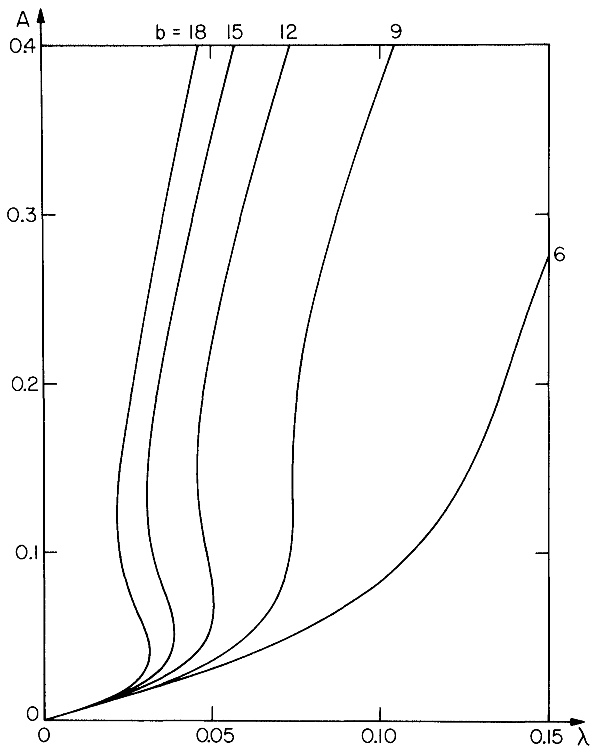

FIG. 2

such a situation is compatible with the theory we shall construct a series of problems where $f(u)$ resembles the Arrhenius reaction rate as a function of temperature.

The Arrhenius reaction rate behaves typically as follows: Starting from a comparatively low value at $u = 0$ it increases to a maximum with a change in the sign of its curvature and then drops rapidly to zero for some finite value of u. In order to obtain a function $f(u)$ with these properties, we use a $\lambda = \lambda(z)$ which is given by

$$(2\lambda)^{1/2} = \frac{2z^{1/2}}{1 + bV_0^{-1}z} - 2z^{1/2}k \ln(1 - V_0^{-1}z), \tag{3.5}$$

where b, V_0, and k are positive constants. With this λ, $f(0) = 1$ for all values of b, V_0, and k according to Theorem 3. The first term in (3.5) has a maximum point at $z = V_0/b$. If k is chosen small enough, the first term is dominant except near $z = V_0$ where the second term goes to infinity. In all the numerical examples in Figs. 3 and 4, $k = 0.01$. Therefore, if b is chosen so large that the maximum of the first term in (3.5) is well below V_0, the function $\lambda = \lambda(z)$ will also have a maximum near $z = V_0/b$. Since the second term goes to infinity as $z \to V_0$, it appears that by choosing k small enough and b large enough it is possible to obtain a function $\lambda = \lambda(z)$ which assumes some of its values two or three times thus giving rise to problems of the form (1.1) with λ's having multiplicity one, two, or three. It follows from Theorem 6 that the function $f(u)$ goes linearly through zero at $u = \varphi(V_0)$. We also note that the choice of a small k assures that $f(u)$ drops steeply towards zero as does the Arrhenius reaction rate.

From (2.8) the function $u = \varphi(V)$ corresponding to $\lambda = \lambda(z)$ given by (3.5) is found to be

$$u = 2\{V_0 b^{-1}[1 - (1 + bV_0^{-1}V)^{-1/2}] - k[V \ln(\tfrac{1}{2}(1 + (1 - V/V_0)^{1/2})) - \tfrac{1}{2}V_0(1 - (1 - V/V_0)^{1/2})^2]\} \tag{3.6}$$

and $f = [\varphi'(V)]^{-1}$ is

$$f = \{(1 + bV_0^{-1}V)^{-3/2} + 2k[(1 - V/V_0)^{-1/2} - 1 - \ln(\tfrac{1}{2}(1 + (1 - V/V_0)^{1/2}))]\}^{-1}. \tag{3.7}$$

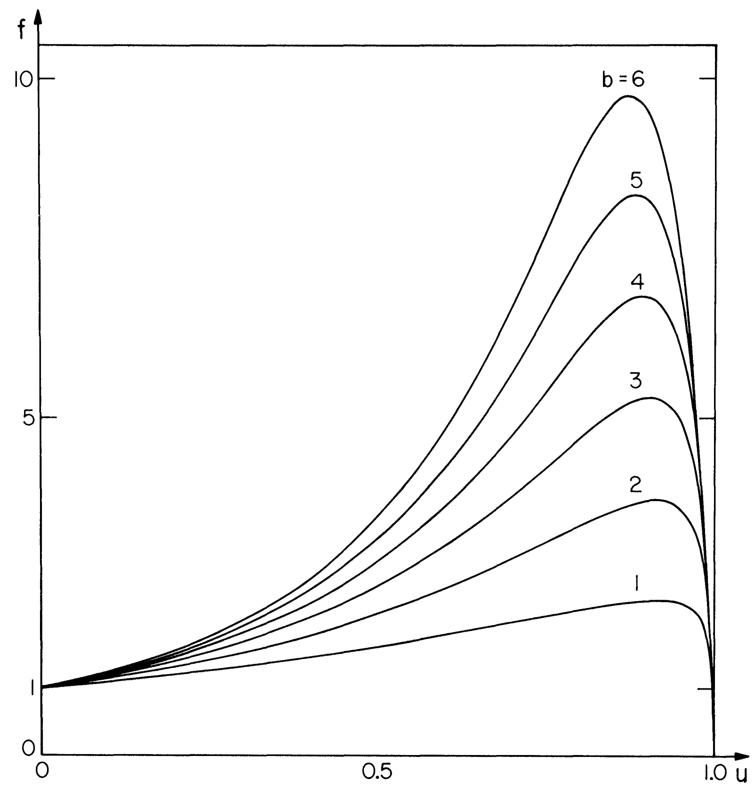

FIG. 3

In order to adjust $f(u)$ so that the zero of f is at $u = 1$ for all values of b and k we choose V_0 to be

$$V_0 = \{2b^{-1}(1 - (1 + b)^{-1/2}) + k(2 \ln 2 + 1)\}^{-1}.$$

In Figs. 3 and 4 the function $f(u)$ and the response curve ($\lambda = \lambda(z)$ versus $A = \varphi(z)$) are plotted for different values of b. In all cases $k = 0.01$. It is seen

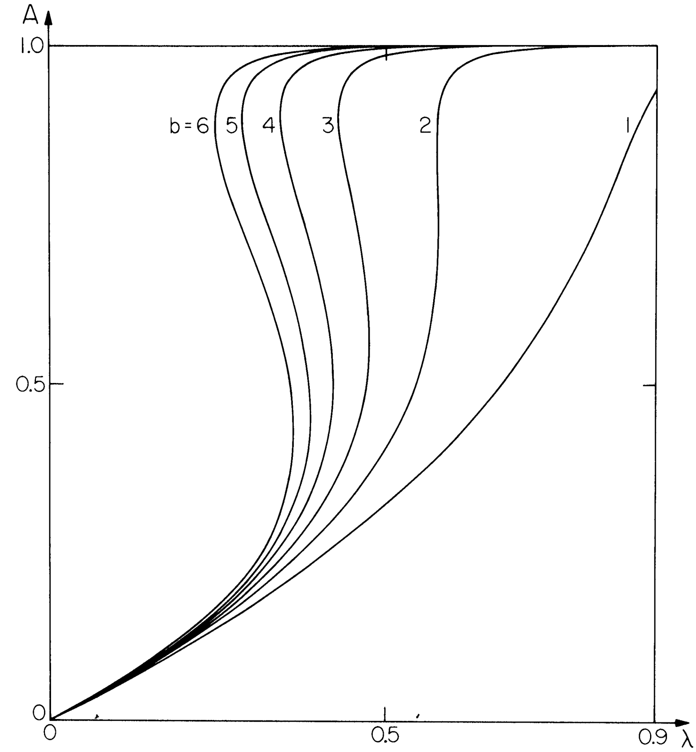

FIG. 4

from Fig. 4 that the bifurcation value of b (i.e., the value separating uniqueness from higher multiplicities) is $b \approx 2$. A result which has important implications in the chemical reactor theory is that the interval of λ's with multiplicity three becomes larger the more pronounced the maximum of the function $f(u)$. In some special cases (corresponding to very large Peclet number in the reactor theory) D. S. Cohen [4] has applied a formal singular perturbation procedure to exploit this pronounced maximum; he has shown, in fact, that in these cases there are three solutions and that the upper and lower of the three solutions are physically stable, a result observed in the experimental work.

As in Example 1 the uniqueness result of Cohen and Laetsch [2] applies here also if the reaction rate $f(u)$ satisfies (3.4). The same discussion is applicable here, and our Figs. 3 and 4 once again indicate that uniqueness is lost when (3.4) is violated.

Example 3. The formulas (2.8) and (2.9) allow us to construct problems of the form (1.1) which have any desired number of solutions for some values of the parameter λ. We shall now construct one with up to four solutions. Its $f(u)$-function satisfies the conditions of Laetsch's theorem (see § 2) except in an interval of u. The example shows to what extent the conditions of that theorem have to be violated in order that more than two solutions exist, and, in particular, it indicates the phenomena occurring for certain nonlinearities arising in problems in nonlinear heat generation [1], [6].

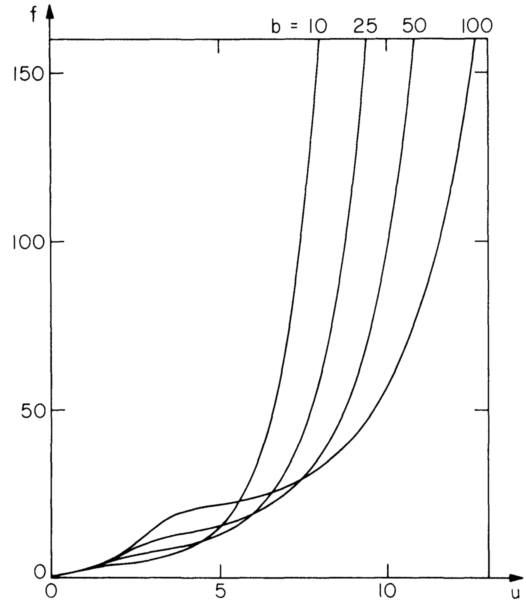

FIG. 5

The function

$$f(u) = e^u, \qquad u \geqq 0, \tag{3.8}$$

satisfies all the conditions of Laetsch's theorem. Its parametric representation, with V as the parameter, is

$$u \equiv \ln(V + 1) \tag{3.9}$$

and

$$f \equiv V + 1. \tag{3.10}$$

The function $\lambda = \lambda_1(z)$ corresponding to (3.8) is

$$\lambda_1(z) = \frac{2}{z + 1}[\ln((z + 1)^{1/2} + z^{1/2})]^2. \tag{3.11}$$

The corresponding response curve ($\lambda = \lambda_1(z)$ versus $A = \varphi(z)$) is shown in Fig. 6 (the curve marked $b = 0$). In accordance with Laetsch's theorem it has one maximum and goes to zero as $A \to \infty$.

Now, consider the function $\lambda = \lambda(z)$ given by

$$(2\lambda)^{1/2} = (2\lambda_1)^{1/2} + \frac{(z/b)^{3/2}}{1 + (z/b)^2}, \tag{3.12}$$

where λ_1 is given by (3.11). The last term in (3.12) has a maximum for $z = 3^{1/2}b$ and goes to zero as $z \to \infty$. Therefore, for small values of b the maximum of the second term on the right-hand side of (3.12) only enhances the maximum of the function $\lambda = \lambda(z)$ stemming from the first term, and thus, the problem will have

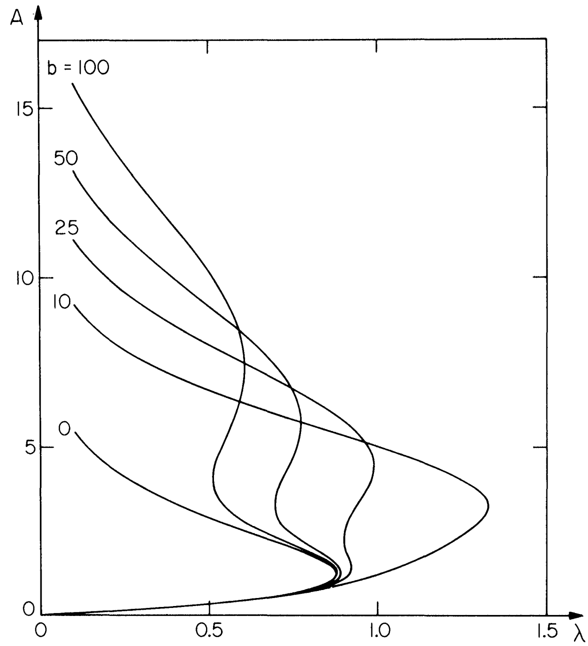

FIG. 6

two, one, or zero positive solutions depending on the value of λ. For larger values of n the maximum of the function $\lambda = \lambda(z)$ arising from the second term is detached from that arising from the first one and the problem will have four, three, two, one, or zero solutions depending on the value of λ.

When λ given by (3.12) is inserted in (2.8) and (2.9) the parametric form of the corresponding function $f(u)$ is found to be

$$u = \ln(V+1) + b^{1/2} - y^{-1}[\tfrac{1}{2}b(1+y)]^{1/2} \tag{3.13}$$

and

$$f = \left\{\frac{1}{V+1} + \frac{V}{(2b)^{3/2}}\cdot\frac{2+y}{y^3(1+y)^{1/2}}\right\}^{-1}, \tag{3.14}$$

where

$$y = (1 + (V/b)^2)^{1/2}. \tag{3.15}$$

The graph of $f(u)$ given by (3.13) and (3.14) and the response curve ($\lambda = \lambda(z)$ versus $A = \varphi(z)$) are shown in Figs. 5 and 6 respectively for $b = 10$, 25, 50, and 100. In all these cases the three conditions of Laetsch's theorem: $f(u) > 0$, $f'(u) > 0$ for $u \geqq 0$ and $f(u)/u \to \infty$ as $u \to \infty$ are satisfied while the fourth, $f''(u) > 0$ for $u \geqq 0$, is not. We note that the interval in which $f''(u) < 0$ becomes larger the larger b is, but that it is present also for $b = 10$, in which case the response curve is the same as if Laetsch's theorem were valid. The example therefore shows that in order that the problem have more than two solutions for some values of λ the conditions of Laetsch's theorem must be violated over a finite interval of u. On the other hand, the case $b = 25$, where there are up to four solutions, shows that only a minute change of the function $f(u)$ is sufficient to introduce the two extra solutions.

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